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LETTER TO THE EDITOR

A simple proof of spin-$\frac{1}{2}$ X–Y inequalities

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Abstract. A simple proof, based on duplicate variables, is presented for various correlation inequalities for the spin-$\frac{1}{2}$ X–Y model. In addition, a conjecture is made that would allow these correlation inequalities to be obtained for the general spin-s X–Y model.

Gallavotti (1971) and Suzuki (1973) have derived a number of correlation inequalities for the spin-$\frac{1}{2}$ X–Y model. The proofs of these inequalities, however, are complicated by commutation difficulties and rely on the use of Trotter's formula. Here we give a simple derivation of these inequalities using duplicate variables.

The duplicate variable method of proving correlation inequalities has been highly successful for Ising models (Sylvester 1976) and vector spin models (Monroe and Pearce 1979), but so far not for quantum models. Although Ginibre (1970) proposed a possible framework for non-commutative models, examples satisfying his general theory have not been found. In contrast, the method presented here works not only for the spin-$\frac{1}{2}$ X–Y model, but has possibilities for extension to other quantum models, in particular, the general spin-s X–Y model. The only inequalities known at present for this general model are comparison inequalities (Pearce 1979).

Let Λ be a finite set of sites. At each site $i \in \Lambda$ locate a spin $\boldsymbol{\sigma}_i = (\sigma_i^x, \sigma_i^y, \sigma_i^z)$, where σ_i^x, σ_i^y, σ_i^z are the usual Pauli spin matrices acting on the vector space $\otimes_{i\in\Lambda}\mathbb{C}^2$. Explicitly, $\sigma_i^z = 1\otimes 1\otimes \ldots \otimes\sigma^z\otimes\ldots\otimes 1$ where the 2×2 Pauli matrix σ^z appears as the ith factor, etc. If we set

$$\sigma_A^z = \prod_{i\in A}\sigma_i^z \qquad \sigma_A^x = \prod_{i\in A}\sigma_i^x, \tag{1}$$

the spin-$\frac{1}{2}$ X–Y model Hamiltonian can be written as

$$H(\boldsymbol{\sigma}) = -\sum_{A\subset\Lambda}(J_A^z\sigma_A^z + J_A^x\sigma_A^x) \tag{2}$$

where J_A^z, J_A^x are interaction parameters.

A duplicate system is formed by associating an additional spin $\bar{\boldsymbol{\sigma}}_i = (\bar{\sigma}_i^x, \bar{\sigma}_i^y, \bar{\sigma}_i^z)$, where $\bar{\sigma}_i^x$, $\bar{\sigma}_i^y$, $\bar{\sigma}_i^z$ are also Pauli spin matrices, to each site $i\in\Lambda$. On the doubled system the symbols σ_i^z, $\bar{\sigma}_i^z$, etc, stand for the matrices $\sigma_i^z\otimes 1$, $1\otimes\sigma_i^z$, etc operating on the vector space $(\otimes_{i\in\Lambda}\mathbb{C}^2)\otimes(\otimes_{i\in\Lambda}\mathbb{C}^2)$. The Hamiltonian for the doubled system is taken to be

$$\mathscr{H}(\boldsymbol{\sigma}, \bar{\boldsymbol{\sigma}}) = H(\boldsymbol{\sigma}) + \bar{H}(\bar{\boldsymbol{\sigma}}) \tag{3}$$

where

$$\bar{H}(\bar{\boldsymbol{\sigma}}) = -\sum_{A\subset\Lambda}(\bar{J}_A^z\bar{\sigma}_A^z + \bar{J}_A^x\bar{\sigma}_A^x). \tag{4}$$

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Here $\bar{\sigma}_A^z$, $\bar{\sigma}_A^x$, are defined similarly to σ_A^z, σ_A^x and $\bar{J}_A^z$, $\bar{J}_A^x$ are interaction parameters which may or may not be the same as J_A^z, J_A^x.

Let P be the (non-commutative) multiplicative cone generated by the variables $\sigma_i^z \pm \bar{\sigma}_i^z$, $\bar{\sigma}_i^x \pm \sigma_i^x$, $i \in \Lambda$, so that fg, $f+g$ and $af \in P$ whenever $f, g \in P$ and $a \geqslant 0$. If $f \in P$ and $-\mathscr{H} \in P$ we assert that

$$\langle f \rangle \equiv \operatorname{Tr} f \exp(-\mathscr{H}) / \operatorname{Tr} \exp(-\mathscr{H}) \geqslant 0, \tag{5}$$

where for convenience we have set the inverse temperature $\beta = 1$. Furthermore, we claim that this remarkable inequality has the following consequences for the spin-$\frac{1}{2}$ X–Y model (2):

$$\langle \sigma_A^z \rangle \geqslant 0 \qquad \langle \sigma_A^z \sigma_B^z \rangle - \langle \sigma_A^z \rangle \langle \sigma_B^z \rangle \geqslant 0 \tag{6}$$

$$\partial \langle \sigma_A^z \rangle / \partial J_B^z \geqslant 0 \qquad \partial \langle \sigma_A^z \rangle / \partial J_B^x \leqslant 0, \tag{7}$$

under the conditions $J_A^z \geqslant 0$, $J_A^x \geqslant 0$ for all $A \subset \Lambda$.

To prove (5) we proceed as follows. Since $\exp h \in P$ whenever $h \in P$, we only need prove $\operatorname{Tr} g \geqslant 0$ for all $g \in P$. Moreover, by linearity of the trace, it suffices to consider the case when g is a product of the variables $(\sigma_i^z \pm \bar{\sigma}_i^z)$, $(\bar{\sigma}_i^x \pm \sigma_i^x)$, $i \in \Lambda$. But now, because the trace factors over the sites, $\operatorname{Tr} g$ will be non-negative if

$$\operatorname{Tr} \prod (\sigma^z \pm \bar{\sigma}^z)(\bar{\sigma}^x \pm \sigma^x) \geqslant 0 \tag{8}$$

where the product sign indicates an arbitrary product of the four factors. The left-hand side of (8) can be written as

$$\sum_{\mu, \bar{\mu} = \pm 1} \langle \mu \bar{\mu} | \prod (\sigma^z \otimes 1 \pm 1 \otimes \sigma^z)(1 \otimes \sigma^x \pm \sigma^x \otimes 1) | \mu \bar{\mu} \rangle \tag{9}$$

where the trace has been evaluated in the product basis $|\mu\bar{\mu}\rangle = |\mu\rangle|\bar{\mu}\rangle$ and the Ising variables $\mu = \pm 1$, etc label the eigenstates of σ^z; $\sigma^z|\mu\rangle = \mu|\mu\rangle$. Inserting intermediate complete sets of states,

$$\sum_{\mu, \bar{\mu} = \pm 1} |\mu\bar{\mu}\rangle\langle\mu\bar{\mu}| \equiv 1 \tag{10}$$

between each factor in the product, it is easily verified that the resulting matrix elements are given by

$$\begin{aligned} &\langle \mu\bar{\mu} | \sigma^z \otimes 1 \pm 1 \otimes \sigma^z | \mu'\bar{\mu}' \rangle = \tfrac{1}{4}(\mu \pm \bar{\mu})(1 + \mu\mu')(\mu\mu' + \bar{\mu}\bar{\mu}') \\ &\langle \mu\bar{\mu} | 1 \otimes \sigma^x + \sigma^x \otimes 1 | \mu'\bar{\mu}' \rangle = \tfrac{1}{4}\mu\mu'(\mu\mu' - \bar{\mu}\bar{\mu}') \\ &\langle \mu\bar{\mu} | 1 \otimes \sigma^x - \sigma^x \otimes 1 | \mu'\bar{\mu}' \rangle = \tfrac{1}{4}(\mu\mu' - \bar{\mu}\bar{\mu}'). \end{aligned} \tag{11}$$

Each of these belongs to the multiplicative convex cone Q generated by all the variables $(\mu \pm \bar{\mu})$, $(\mu' \pm \bar{\mu}')$, $(\mu'' \pm \bar{\mu}'')$, etc. Thus the required result (8) follows from the Ising duplicate variable inequality

$$\sum_{\mu, \bar{\mu} = \pm 1} \sum_{\mu', \bar{\mu}' = \pm 1} \dots f \geqslant 0 \tag{12}$$

which holds (Sylvester 1976) for all $f \in Q$.

To obtain inequalities (6) and (7) from (5), notice that $\sigma_A^z \pm \bar{\sigma}_A^z$, $\bar{\sigma}_A^x \pm \sigma_A^x \in P$, and hence $-\mathscr{H} \in P$ provided $|\bar{J}_A^z| \leqslant J_A^z$ and $|J_A^x| \leqslant \bar{J}_A^x$ for all $A \subset \Lambda$. Now to obtain the inequalities (6) set $\bar{J}_A^z = J_A^z$, $\bar{J}_A^x = J_A^x$ for all $A \subset \Lambda$ and choose $f = \sigma_A^z \in P$ and $f =$

$\sigma^z_A(\sigma^z_B - \bar{\sigma}^z_B) \in P$ in (5). Alternatively, if we require $J^z_A > \bar{J}^z_A \geq 0$, $\bar{J}^x_A > J^x_A \geq 0$ for all $A \subset \Lambda$ and choose $f = \sigma^z_A - \bar{\sigma}^z_A \in P$ in (5) we find that

$$
\begin{aligned}
&(\langle\sigma^z_A\rangle - \langle\bar{\sigma}^z_A\rangle)/(J^z_B - \bar{J}^z_B) \geq 0 \\
&(\langle\sigma^z_A\rangle - \langle\bar{\sigma}^z_A\rangle)/(J^x_B - \bar{J}^x_B) \leq 0.
\end{aligned}
\tag{13}
$$

Now taking the limits $\bar{J}^z_A \to J^z_{A^-}$, $\bar{J}^x_A \to J^x_{A^+}$ for all $A \subset \Lambda$ we obtain the inequalities (7).

In conclusion, we remark that it seems reasonable to conjecture that the crucial inequality (8) holds when $\boldsymbol{\sigma}$ is a general spin-s angular momentum operator. In this case the inequalities (6) and (7) would hold for the spin-s X–Y model. We have been unable either to prove this conjecture or to find a simple counterexample. It should be observed, however, that the inequality (8) does hold (Monroe and Pearce 1979) in the classical limit $s \to \infty$ (Lieb 1973) and yields the known correlation inequalities (Kunz *et al* 1976) in this case.

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